CLARSTA: A Random Subspace Trust-region Algorithm for Convex-constrained Derivative-free Optimization

Yiwen Chen

Department of Mathematics
University of British Columbia

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Based on joint work with Warren Hare and Amy Wiebe
Includes joint work with Lindon Roberts

High-dimensional model-based DFO

We consider

$$\min_{x \in \Omega} f(x)$$

where $\Omega \subseteq \mathbb{R}^n$ and f is given by a black box:

Model-based DFO methods:

- Construct surrogate models to approximate objective function

Issue in high dimensions: full-space model cost $\sim \mathcal{O}(n)$ or $\mathcal{O}(n^2)$

Idea:

$$\mathbb{R}^n \quad \rightsquigarrow \quad x_k + \mathrm{col}(Q_k) \;\text{ where }\; Q_k \in \mathbb{R}^{n \times p},\; Q_k^\top Q_k = I_p,\; p \ll n$$

CLARSTA at a glance

Main story

CLARSTA is a random-subspace trust-region framework for

$$\min_{x \in C} f(x)$$

where C is convex, closed, and has a nonempty interior

From QARSTA to CLARSTA

for $k = 0, 1, \ldots$ **do**

Define an affine subspace $x_k + \mathrm{col}(Q_k)$ by selecting $Q_k \in \mathbb{R}^{n \times p}$

Construct a model $\widehat{m}_k$ in $\mathbb{R}^p$

Approximately solve the trust-region subproblem in $\mathbb{R}^p$:

$$\widehat{s}_k \approx \operatorname*{argmin}_{\widehat{s} \in \mathbb{R}^p} \widehat{m}_k(\widehat{s}), \quad s.t. \ \|\widehat{s}\| \leq \Delta_k$$

and calculate the corresponding step $s_k \in \mathbb{R}^n$

Evaluate $f(x_k + s_k)$ and calculate descent ratio

$$\rho_k = \frac{f(x_k) - f(x_k + s_k)}{\widehat{m}_k(\mathbb{0}) - \widehat{m}_k(\widehat{s}_k)} = \frac{true \ \ decrease}{predicted \ \ decrease}$$

Accept/reject step based on ρ_k and update trust region radius

General framework of CLARSTA

for $k = 0, 1, \dots$ **do**

Define an affine subspace $x_k + \mathrm{col}(Q_k)$ by *selecting* $Q_k \in \mathbb{R}^{n \times p}$

Construct a model $\widehat{m}_k$ in $\mathbb{R}^p$

Approximately solve the trust-region subproblem in $\mathbb{R}^p$:

$$\widehat{s}_k \approx \underset{\widehat{s} \in Q_k^\top (C - x_k)}{\operatorname{argmin}} \quad \widehat{m}_k(\widehat{s}), \quad s.t. \ \|\widehat{s}\| \leq \Delta_k$$

and calculate the corresponding feasible step $s_k \in \mathbb{R}^n$

Evaluate $f(x_k + s_k)$ and calculate descent ratio

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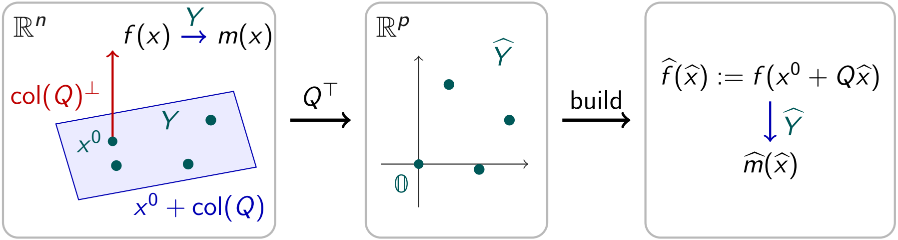
Subspace vs. full-space models



Lift and equivalence (Chen 2026)

For $\mathcal{X} \in \{\mathrm{MN}, \mathrm{MFN}, \ldots\}$, we have

$$\nabla m_{\mathcal{X}}(x^0) = Q \nabla \widehat{m}_{\mathcal{X}}(\mathbb{0}), \quad \nabla^2 m_{\mathcal{X}}(x^0) = Q \nabla^2 \widehat{m}_{\mathcal{X}}(\mathbb{0}) Q^\top,$$

and

$$m_{\mathcal{X}}(x) = \widehat{m}_{\mathcal{X}}(\widehat{x}), \quad \text{for all } x \in \left(x^0 + Q\widehat{x}\right) + \mathrm{col}(Q)^\perp$$

From fully linear to (C, Q)-fully linear

Model class	**Accuracy region**	**How to build it**
Classic fully linear	$B(x^0; \Delta)$	Full-space linear interpolation
Q-fully linear	$B(x^0; \Delta) \cap \left(x^0 + \mathrm{col}(Q)\right)$	Subspace linear interpolation
(C, Q)-fully linear	$B(x^0; \Delta) \cap \mathrm{proj}_{x^0 + \mathrm{col}(Q)}(C)$	Subspace linear interpolation with *feasible geometry*

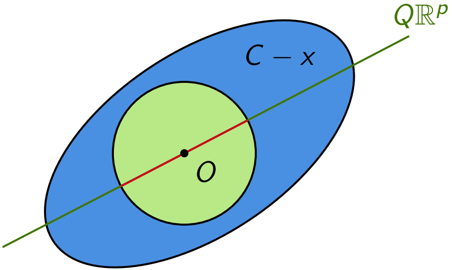
(C, Q)-fully linear models

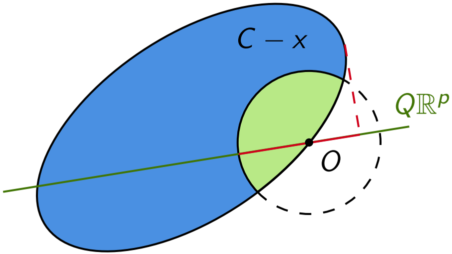
Definition (Chen-Hare-Wiebe 2025)

Let $Q \in \mathbb{R}^{n \times p}$ consist of p orthonormal columns
A model $\widehat{m} : \mathbb{R}^p \to \mathbb{R}$ is (C, Q)-fully linear in $B(x, \Delta)$ if there exist $\kappa_f, \kappa_g > 0$ s.t. for all $\widehat{s} \in Q^\top (C - x)$ with $\|\widehat{s}\| \le \Delta$,

$$|f(x + Q\widehat{s}) - \widehat{m}(\widehat{s})| \le \kappa_f \Delta^2$$

$$\max_{d \in Q^\top (C - x), \|d\| \le 1} \left| \left(Q^\top \nabla f(x + Q\widehat{s}) - \nabla \widehat{m}(\widehat{s}) \right)^\top d \right| \le \kappa_g \Delta$$





Classical geometry measures can blow up

Problem (Larson-Menickelly-Wild 2019)

"For a fixed value of κ, it may be impossible to obtain a κ-fully linear model using only feasible points"

Constrained example (Hough-Roberts 2022)

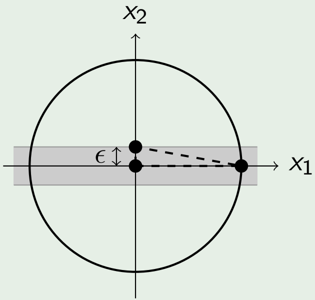
Consider

$$C = \{(x_1, x_2) : |x_2| \leq \epsilon\} \subseteq \mathbb{R}^2$$

and sample set

$$Y = \{\mathbb{0},\ (1,0),\ (0,\epsilon)\} \subseteq B(\mathbb{0}, 1) \cap C$$

Classical geometry measures are $\sim \epsilon^{-1}$



Feasible sample set geometry measure

Let D consist of sample directions and write $D = QR$, where $Q^\top Q = I_p$

Feasible geometry measure: matrix spectral norm w.r.t. $Q^\top(C - x^0)$

For any matrix $M \in \mathbb{R}^{m \times p}$, define

$$\|M\|_{x^0,C,D} := \frac{1}{\min\{\overline{\mathrm{diam}}(R), 1\}} \max_{\substack{w \in Q^\top(C - x^0) \\ \|w\| \leq \min\{\overline{\mathrm{diam}}(R), 1\}}} \|Mw\|$$

where $\overline{\mathrm{diam}}(R) := \max_{r_i \in R} \|r_i\|$

Note: If $C = \mathbb{R}^n$, this reduces to the spectral norm: $\|M\|_{x^0,C,D} = \|M\|$

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Note: If $C = \mathbb{R}^n$, this reduces to the spectral norm: $\|M\|_{x^0,C,D} = \|M\|$

Why this fixes the skinny-feasible-set issue

For $C = \{(x_1, x_2) : |x_2| \le \epsilon\}$, $x^0 = \mathbb{0}$, $D = I_2$, and $M = \mathrm{Diag}(1, \epsilon^{-1})$,

$$\|M\|_{\mathbb{0},C,I_2} = \max_{|w_2| \le \epsilon, \|w\| \le 1} \sqrt{w_1^2 + \epsilon^{-2} w_2^2} = \sqrt{2 - \epsilon^2} \le \sqrt{2}$$

Constructing (C, Q)-fully linear models

Let D consist of sample directions and write $D = QR$, where $Q^\top Q = I_p$

Theorem

If $f \in \mathcal{C}^{1+}$, D has full column rank, and

$$\left\| \overline{R}^{-1} \right\|_{x^0, C, D} \quad \text{is uniformly bounded, where } \overline{R} := R / \overline{\mathrm{diam}}(R),$$

then the subspace linear interpolation model $\widehat{m}$ is (C, Q)-fully linear in $B(x^0; \overline{\mathrm{diam}}(R))$

α-well-aligned matrices (unconstrained version)

Let $x + \mathrm{col}(D)$ be the affine subspace

Definition (Cartis-Roberts 2023)

Let $\alpha \in (0,1)$. We say that $D \in \mathbb{R}^{n \times p}$ is α-well-aligned for f at x if

$$\left\| D^\top \nabla f(x) \right\| \geq \alpha \left\| \nabla f(x) \right\|$$

Theorem (Dzahini-Wild 2024)

(Idea: Johnson–Lindenstrauss Lemma)

Suppose $\alpha, \delta \in (0,1)$, $p \geq 4(1-\alpha)^{-2} \ln(1/\delta)$, *and i.i.d.* $D_{ij} \sim \mathcal{N}(0, 1/p)$. *Then,*

$$\mathbb{P}\left[D \text{ is } \alpha\text{-well-aligned for } f \text{ at } x\right] \geq 1 - \delta$$

First-order criticality measure

Definition (Conn-Gould-Toint 2000)

First-order criticality measure for convex-constrained optimization

$$\pi^f(x) := \left| \min_{\substack{x+d \in C \\ \|d\| \leq 1}} \nabla f(x)^\top d \right|$$

Interpretation

$\pi^f(x)$ measures the maximum first-order feasible decrease at x

α-well-aligned matrices (convex-constrained version)

Let $x + \mathrm{col}(D)$ be the affine subspace and decompose $D = QR$

Definition (Chen-Hare-Wiebe 2025)

Let $\alpha \in (0, 1)$. We say that $D \in \mathbb{R}^{n \times p}$ is α-well-aligned for f and C at x if

$$\left| \min_{\substack{d \in C - x \\ \|d\| \leq 1}} \nabla f(x)^\top Q Q^\top d \right| \geq \alpha \pi^f(x)$$

Theorem (Chen-Hare-Wiebe 2025)

(Idea: Concentration of measure on the Grassmannian)

Suppose $\beta \in (0, 1)$, $p \in (0, n]$, and i.i.d. $D_{ij} \sim \mathcal{N}(0, 1)$. Then,

$$\mathbb{P}\left[D \text{ is } \left(\frac{p}{n}\beta\right)\text{-well-aligned for } f \text{ and } C \text{ at } x\right]$$
$$\geq \text{complicated stuff that depends on } n,\ p,\ \beta,\ \pi^f(x), \text{ and } \|\nabla f(x)\|$$

Convergence and complexity results

Let $\epsilon > 0$, (UC)=UnConstrained, and (CC)=Convex-Constrained

- (UC) $\mathbb{P}\left[\min_{k \le K} \|\nabla f(x_k)\| < \epsilon\right] \ge 1 - e^{-C(K+1)}$

  (CC) $\mathbb{P}\left[\min_{k \le K} \pi^f(x_k) < \epsilon\right] \ge 1 - e^{-C(\epsilon)(K+1)}$

- (UC) $\mathbb{P}\left[\inf_{k \ge 0} \|\nabla f(x_k)\| = 0\right] = 1$

  (CC) $\mathbb{P}\left[\inf_{k \ge 0} \pi^f(x_k) = 0\right] = 1$

- (UC) $\mathbb{E}\left[\min\{k \ge 0 : \|\nabla f(x_k)\| < \epsilon\}\right] = \mathcal{O}(\epsilon^{-2})$

  (CC) $\mathbb{E}\left[\min\left\{k \ge 0 : \pi^f(x_k) < \epsilon\right\}\right] = \mathcal{O}(\epsilon^{-4})$

(UC) from [Cartis, Roberts, 2023]; (CC) from [Chen, Hare, Wiebe, 2025]

Comparison of runtime to reach f^*

Steps:

1. Run $\mathrm{CLARSTA}$ for $100(n+1)$ fevals and denote the result by f^*
2. Run COBYLA until f^* is reached or $100(n+1)$ fevals or 10^5 seconds are required

Results when $n = 1000$:

- In some problems, COBYLA uses less fevals to reach f^*
- COBYLA requires more than 1000 times more time than $\mathrm{CLARSTA}$ to reach f^* (1min vs. 17h)

Results when $n = 10000$:

- $\mathrm{CLARSTA}$ reaches f^* in 30 minutes
- COBYLA never reaches f^* in 10^5 seconds ($\sim$ 28h)

Toward quadratic CLARSTA

Quadratic models

Minimum norm (MN), minimum Frobenius norm (MFN), quadratic generalized simplex derivative (QS) models, etc.

Fully linear bounds

MN and MFN bounds were known previously
In [Chen-Hare-Roberts 2026], we

- remove the bounded model-Hessian assumption for MN
- give the first fully linear bounds for QS

Directional Hessian accuracy

With a mild sample set structure, all three models capture fully quadratic accuracy along sample directions

Directional Hessian accuracy

Hessian error ranges from $\mathcal{O}(1)$ to $\mathcal{O}(\Delta_Y)$ (Chen-Hare-Roberts 2026)

Suppose that $f \in \mathcal{C}^{2+}$,

$$Y = \left\{x^0, x^0 \pm d_i : i \in \{1, \ldots, p\}\right\}, \quad \Delta_Y := \max_i \|d_i\|, \quad \overline{d} := d/\|d\|$$

For $\mathcal{X} \in \{\mathrm{MN}, \mathrm{MFN}, \mathrm{QS}\}$,

$$\left|\overline{d_i}^\top \left(\nabla^2_{\mathcal{X}} f(x^0; Y) - \nabla^2 f(x^0)\right) \overline{d_i}\right| = \mathcal{O}(\Delta_Y)$$

Directional Hessian accuracy

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Suppose that $f \in \mathcal{C}^{2+}$,

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For $\mathcal{X} \in \{\mathrm{MN}, \mathrm{MFN}, \mathrm{QS}\}$,

$$\left|\overline{d_i}^\top \left(\nabla^2_{\mathcal{X}} f(x^0; Y) - \nabla^2 f(x^0)\right) \overline{d_i}\right| = \mathcal{O}(\Delta_Y)$$

In general, for any non-zero $d \in \mathrm{col}(D)$, and $v := D^\dagger d$,

$$\begin{aligned}&\left|\overline{d}^\top \left(\nabla^2_{\mathcal{X}} f(x^0; Y) - \nabla^2 f(x^0)\right) \overline{d}\right| \\ &= \mathcal{O}\left(\frac{\|v\|_1^2 - \|v\|_\infty^2}{\|v\|^2}\right) + \mathcal{O}\left(\frac{\|v\|_1^2 - \|v\|_\infty^2}{\|v\|^2}\Delta_Y\right) + \mathcal{O}(\Delta_Y)\end{aligned}$$

Summary

CLARSTA is a new approach to large-scale convex-constrained DFO:

- Extends unconstrained framework using feasible subspace geometry and (C, Q)-fully linear models
- Samples random subspaces that are sufficiently aligned with feasible descent directions, yielding convergence and complexity bounds
- Demonstrates competitive high-dimensional performance on problems with up to 10000 variables

Ongoing: incorporate quadratic models into CLARSTA :)

Thank you

- Yiwen Chen. *Relationships between full-space and subspace quadratic interpolation models and simplex derivatives*. 2026. arXiv: 2602.10374
- Yiwen Chen, Warren Hare, and Lindon Roberts. *Accuracy and Relationships of Quadratic Models in Derivative-free Optimization*. 2026. arXiv: 2605.12819
- Yiwen Chen, Warren Hare, and Amy Wiebe. *CLARSTA: A random subspace trust-region algorithm for convex-constrained derivative-free optimization*. 2025. arXiv: 2506.20335

Email: yiwchen@student.ubc.ca