# Determining inscribability of polytopes via rank minimization based on slack matrices

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# Outline


1. Introduction
2. Characterizing inscribability using slack matrices
3. An SDP formulation
4. Numerical experiments
5. Summary

# Inscribability of a polytope

Given a polytope, does there exist a combinatorially equivalent* polytope with all vertices on the sphere?

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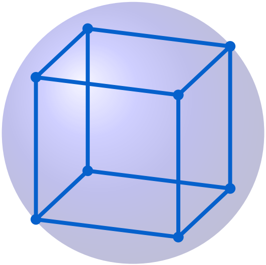
*The face lattices of the two polytopes are isomorphic

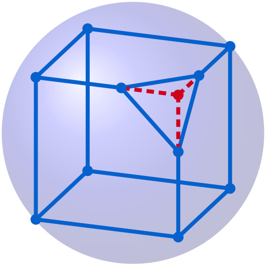
# Inscribability of a polytope

Given a polytope, does there exist a combinatorially equivalent* polytope with all vertices on the sphere?

Yes! $\Rightarrow$ Inscribable



No! $\Rightarrow$ Non-inscribable



*The face lattices of the two polytopes are isomorphic

# The problem

For 3-polytopes:

- [Steiner, 1832]: Are all 3-polytopes inscribable?
- [Steinitz, 1928]: No :(
- [Rivin, 1996]: A complete characterization of inscribable 3-polytopes

# The problem

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For $d$-polytopes where $d > 3$:

- [Padrol, Ziegler, 2016]: Strong necessary and sufficient conditions?
- [Firsching, 2017]: Solving a nonlinear system with $nd$ variables, $\binom{n}{d+1}$ inequalities of degree $d+1$, and $n$ equalities of degree 2, where $n =$ number of vertices

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# Slack matrix

Suppose $P$ is a $d$-polytope such that

- $P = \mathrm{conv}\{v_1, \ldots, v_n\}$
- $P = \{x \in \mathbb{R}^d : c_j - h_j^\top x \geq 0, j = 1, \ldots, m\}$

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A slack matrix $S_P \in \mathbb{R}^{n \times m}$ of $P$ is given by

$$S_P = \begin{bmatrix} & \vdots & \\ \cdots & c_j - h_j^\top v_i & \cdots \\ & \vdots & \end{bmatrix}$$

Note: $\mathrm{rank}(S_P) = d + 1$ [Gouveia et al., 2013] and

$$(S_P)_{ij} \begin{cases} = 0, & \text{if } v_i \text{ is on facet } j \\ > 0, & \text{if } v_i \text{ is not on facet } j \end{cases}$$

Fact: We can WLOG suppose that all $c_j = 1$

# An observation on inscribable polytopes

Suppose $P$ is inscribed in a sphere
Denote $V = [v_1 \cdots v_n]$, $H = [h_1 \cdots h_m]$, and

$$W = \begin{bmatrix} 1 & \mathbb{0}_d^\top \\ \mathbb{1}_n & V^\top \\ \mathbb{1}_m & -H^\top \end{bmatrix} \in \mathbb{R}^{(n+m+1)\times(d+1)}$$

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Then

$$\begin{aligned} X &= WW^\top \\ &= \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & \mathbb{1}_{n\times n} + V^\top V & \mathbb{1}_{n\times m} - V^\top H \\ \mathbb{1}_m & \mathbb{1}_{m\times n} - H^\top V & \mathbb{1}_{m\times m} + H^\top H \end{bmatrix} \succcurlyeq 0 \end{aligned}$$

satisfies $\mathrm{diag}(A) = \mathrm{const.}$, $S \geq 0$, $S$ has the same support as slack matrices of $P$, and $\mathrm{rank}(X) = d + 1$

# Characterizing inscribability using slack matrices

Theorem. A $d$-polytope $P$ is inscribable if and only if there exists

$$X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0$$

such that

- $\mathrm{diag}(A) = \mathrm{const.}$
- $S \geq 0$
- $S$ has the same support as slack matrices of $P$
- $\mathrm{rank}(X) = d + 1$

# Determining inscribability via a min-rank problem

Let $I^z = \{(i,j) : (S_P)_{ij} = 0\}$
Inscribability can be determined by the following min-rank problem:

$$\begin{aligned} \min_{X} \quad & \mathrm{rank}(X) \\ s.t. \quad & X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0 \\ & S_{ij} = 0, \text{ if } (i,j) \in I^z \\ & S_{ij} > 0, \text{ if } (i,j) \notin I^z \\ & A_{ii} = 2, i = 1, \ldots, n \end{aligned}$$

Note: The minimum of this problem is no less than $d + 1$

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# An SDP formulation

Instead of solving the min-rank problem directly, we consider the following SDP problem:

$$
\begin{aligned}
\min_{X} \quad & f_p = \mathrm{tr}(X) - \sum_{(i,j)\notin I^z} \lambda_{ij} S_{ij} \\
s.t. \quad & X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0 \\
& S_{ij} = 0, \text{ if } (i,j) \in I^z \\
& A_{ii} = 2, i = 1, \ldots, n
\end{aligned}
\tag{P}
$$

where $\lambda_{ij}$ are some positive weights

# Dual problem

The dual problem of (P) is

$$\begin{aligned} \max_{u,w} \quad & f_d = m + n + \sum_{\substack{1\le i\le n \\ 1\le j\le m}} M_{ij} - \sum_{i=1}^{n} u_i + 1 \\ s.t. \quad & \begin{bmatrix} I_n + \operatorname{diag}(u) & \frac{1}{2}M \\ \frac{1}{2}M^\top & I_m \end{bmatrix} \succcurlyeq 0 \end{aligned} \tag{D}$$

where

$$M_{ij} = \begin{cases} -\lambda_{ij}, & \text{if } (i,j) \notin I^z \\ w_k \text{ that corresponds to } S_{ij}, & \text{if } (i,j) \in I^z \end{cases}$$

# When the SDP is accurate?

For an inscription $(A^*, B^*, S^*)$, we want to find dual variables $(u^*, w^*)$ s.t.

$$\begin{cases} \begin{bmatrix} I_n + \mathrm{diag}(u^*) & \frac{1}{2}M \\ \frac{1}{2}M^\top & I_m \end{bmatrix} \succcurlyeq 0 \\ f_p^* = f_d^* \end{cases}$$

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If we assume that the inscription is facet transitive[†], centered at the origin, every facet has $k$ vertices, and all

$$u_i^* = \overline{u}, \quad w_i^* = \overline{w}, \quad \lambda_{ij} = \overline{\lambda}$$

then the two conditions are simplified to

$$\begin{cases} \lambda_{\max}(MM^\top) \leq 4 + 4\overline{u} \\ n(1+\overline{u}) + m\|h_1\|^2 = (\overline{\lambda} + \overline{w})km \end{cases}$$

[†]There are rigid linear transformations that send the polytope to itself and send any of its facets to any other of its facets

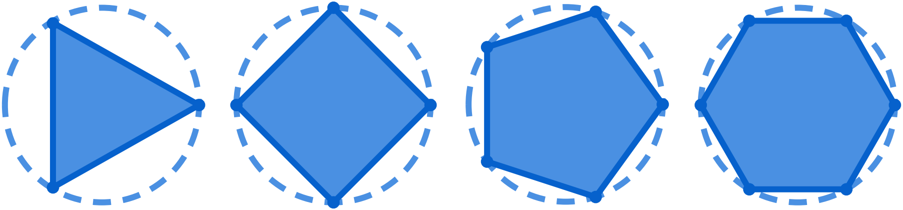
## Examples where the SDP is accurate

# Example: $n$-gons

For $n$-gons, we have $n = m \geq 3$, $d = 2$, and $k = 2$
Consider the inscription:

$$v_i = \left[\cos \frac{2(i-1)\pi}{n} \;\; \sin \frac{2(i-1)\pi}{n}\right]^\top$$

$$h_j = \frac{1}{\cos \frac{\pi}{n}} \left[\cos \frac{(2j-1)\pi}{n} \;\; \sin \frac{(2j-1)\pi}{n}\right]^\top$$



# Example: $n$-gons

Goal: Find $\overline{\lambda}$, $\overline{u}$, and $\overline{w}$ such that

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Notice that

$$MM^\top = \begin{bmatrix} a & b & c & \cdots & b \\ b & a & b & \cdots & c \\ c & b & a & \cdots & c \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix}$$

where $a = (n-2)\overline{\lambda}^2 + 2\overline{w}^2$, $b = (n-3)\overline{\lambda}^2 + \overline{w}^2 - 2\overline{\lambda}\overline{w}$, $c = (n-4)\overline{\lambda}^2 - 4\overline{\lambda}\overline{w}$

# Example: $n$-gons

Set

$$\overline{\lambda} = \frac{2}{n}\sec^2\frac{\pi}{n}, \quad \overline{u} = \tan^2\frac{\pi}{n}, \quad \overline{w} = \frac{n-2}{n}\sec^2\frac{\pi}{n}$$

Then

$$\begin{cases} \lambda_{\max}(MM^\top) = 4\sec^2\frac{\pi}{n} = 4 + 4\overline{u} \\ n(1+\overline{u}) + m\|h_1\|^2 = 2n\sec^2\frac{\pi}{n} = (\overline{\lambda} + \overline{w})km \end{cases}$$

# Example: $n$-gons

Set

$$\overline{\lambda} = \frac{2}{n} \sec^2 \frac{\pi}{n}, \quad \overline{u} = \tan^2 \frac{\pi}{n}, \quad \overline{w} = \frac{n-2}{n} \sec^2 \frac{\pi}{n}$$

Then

$$\begin{cases} \lambda_{\max}(MM^\top) = 4 \sec^2 \frac{\pi}{n} = 4 + 4\overline{u} \\ n(1+\overline{u}) + m\|h_1\|^2 = 2n \sec^2 \frac{\pi}{n} = (\overline{\lambda} + \overline{w})km \end{cases}$$

Theorem. For all $n \geq 3$ and $\lambda_{ij} = \overline{\lambda} = \frac{2}{n} \sec^2 \frac{\pi}{n}, (i,j) \notin I^z$, the SDP has an optimal solution of rank 3 that certifies inscribability of the $n$-gon

# More examples

For $d$-simplices, $d$-cubes, and $d$-crosspolytopes, the SDP also solves the inscribability problem

In particular, solving (P) with the following weights gives an inscription:

$$
\begin{aligned}
(d\text{-simplex}) \quad & \lambda_{ij} = \overline{\lambda} = \frac{2d^2}{d+1}, \quad (i,j) \notin I^z \\
(d\text{-cube}) \quad & \lambda_{ij} = \overline{\lambda} = d2^{1-d}, \quad (i,j) \notin I^z \\
(d\text{-crosspolytope}) \quad & \lambda_{ij} = \overline{\lambda} = 1, \quad (i,j) \notin I^z
\end{aligned}
$$

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# Alternating projection

Recall:

$$\min_{X} \quad \mathrm{rank}(X) \quad s.t. \quad X = \begin{bmatrix} 1 & \mathbb{1}_n^\top & \mathbb{1}_m^\top \\ \mathbb{1}_n & A & S \\ \mathbb{1}_m & S^\top & B \end{bmatrix} \succcurlyeq 0$$

$$S_{ij} = 0, \text{ if } (i,j) \in I^z$$

$$S_{ij} > 0, \text{ if } (i,j) \notin I^z$$

$$A_{ii} = 2, i = 1, \ldots, n$$

- Alternating projection (AP):
  Project $X_k$ between rank $d+1$ cone and feasible set $\Omega$
- Simplified alternating projection (SAP):
  Replace the projection onto $\Omega$ with forcing $X_k$ to have correct constants on correct positions

# Test settings

Algorithms:

- Solving the SDP formulation
- AP (use SDP solution as starting point)
- SAP (use SDP solution as starting point)

Test set: 100 random inscribable simplicial $d$-polytopes with $n$ vertices where $8 \leq n \leq 10$ and $5 \leq d \leq 8$

# Tuning $\lambda_{ij}$ for SDP

- $\lambda^c$: From examples where the SDP is accurate $\lambda_{ij} = \lambda^c = \frac{2d}{n}$
- $\lambda^h$: Heuristic

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Set $\lambda_{ij} = \lambda_{ij}^{\text{init}}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$

**while** $\max\{\lambda_{ij} : i = 1, \ldots, n, j = 1, \ldots, m\} \leq \lambda^{\max}$ **do**

    Solve SDP with current $\lambda_{ij}$

    **if** *SDP solution gives an inscription* **then**

        Return

    **else**

        Set $\lambda_{ij} = \lambda^{\text{inc}} \lambda_{ij}$ for each wrong facet

    **end**

**end**

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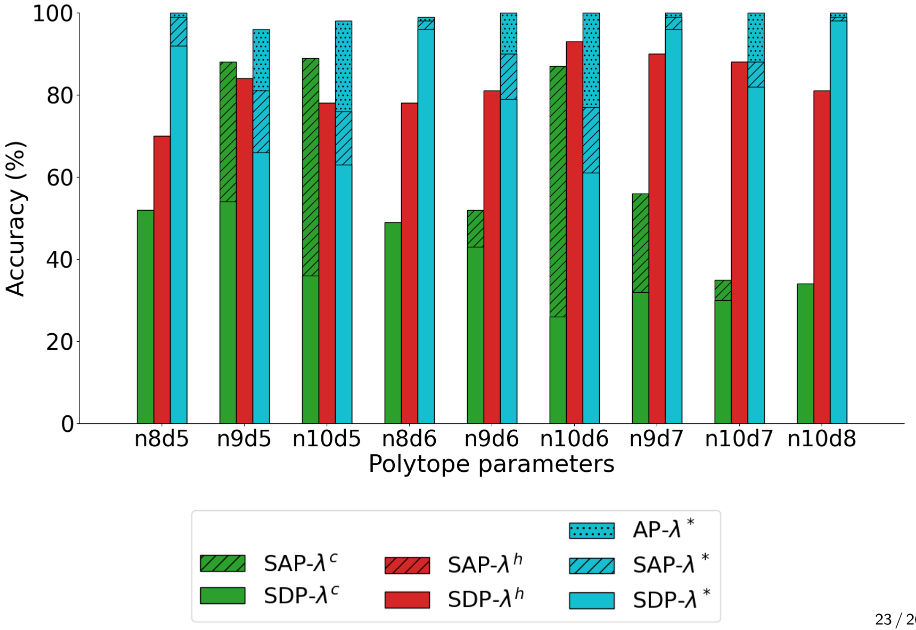
- $\lambda^*$: Minimizing the duality gap for the known inscription

# Results



100
80
60
40
20
0
Accuracy (%)
n8d5
n9d5
n10d5
n8d6
n9d6
n10d6
n9d7
n10d7
n10d8
Polytope parameters
SAP-λ^c
SDP-λ^c
SAP-λ^h
SDP-λ^h
AP-λ*
SAP-λ*
SDP-λ*

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# Summary

In this research, we

- Characterized the inscribability problem as a min-rank problem
- Proposed an SDP approximation for the min-rank problem and proved it is accurate for certain classes of polytopes
- Provided and compared three algorithms, demonstrating our SDP approximation's accuracy, efficiency, and potential in high dimensions

Future work:

- Better heuristics for tuning $\lambda_{ij}$
- Effective and efficient methods for the min-rank problem
- Non-inscribability certificates

# Thank you

Yiwen Chen, João Gouveia, Warren Hare, and Amy Wiebe. *Determining inscribability of polytopes via rank minimization based on slack matrices*. 2025. URL: https://arxiv.org/abs/2502.01878.